Bipartite maximally entangled nonorthorgonal states

Xiaoguang Wang
Institute of Physics and Astronomy, Aarhus University, DK-8000, Aarhus C, Denmark, and
Institute for Scientific Interchange (ISI) Foundation, Viale Settimio Severo 65, I-10133 Torino, Italy
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We give conditions under which general bipartite entangled nonorthogonal states become maximally entangled states. By the conditions we construct a large class of entangled nonorthogonal states with exact one ebit of entanglement in both bipartite and multipartite systems. One remarkable property is that the amount of entanglement in this class of states is independent on the parameters involved in the states. Finally we discuss how to generate the bipartite maximally entangled nonorthogonal states.

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I. INTRODUCTION

Quantum entanglement has generated much interest in the quantum information processing such as quantum teleportation [1], superdense coding [2], quantum key distribution [3], and telecoloning [4]. The entangled orthogonal states receive much attention in the study of quantum entanglement. However the entangled nonorthogonal states also play an important role in the quantum cryptography [5] and quantum information processing [6]. Bosonic entangled coherent states (ECS) [7] and su(2) and su(1,1) ECS [8] are typical examples of entangled nonorthogonal states.

The coherent state can be used to encode quantum information on continuous variables [9] and several schemes [10–12] have been proposed for realizing quantum computation. The overlap $\langle\alpha|-\alpha\rangle$ of two coherent states $|\pm\alpha\rangle$ of π difference is $\exp(-2|\alpha|^2)$, which decreases exponentially with α. Then we can identify the coherent states $|\pm\alpha\rangle$ with large α as basis states of a logical qubit:

$$|0\rangle_L = |\alpha\rangle,\ |1\rangle_L = |-\alpha\rangle. \tag{1}$$

One can also use Schrödinger cat states [13], the even and odd coherent states $|\alpha\rangle_\pm = (|\alpha\rangle \pm |-\alpha\rangle)/\sqrt{2(1\pm e^{-2|\alpha|^2})}$ to encode a qubit [14,15] and they are exactly orthogonal. However these states are extremely sensitive to photon loss.

Now we consider the bosonic ECS [7],

$$|\alpha;\alpha\rangle = \frac{1}{\sqrt{2(1-e^{-4|\alpha|^2})}}\left(|\alpha\rangle\otimes|\alpha\rangle - |-\alpha\rangle\otimes|-\alpha\rangle\right), \tag{2}$$

which can be produced by using a 50/50 beam splitter. If $|\alpha|$ is large, the ECS is considered as a state of two logical qubits (1):

$$|\alpha;\alpha\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_L\otimes|0\rangle_L - |1\rangle_L\otimes|1\rangle_L\right), \tag{3}$$

which is obviously a maximally entangled state (MES), the singlet state.

More superisingly it is found that the ECS $|\alpha;\alpha\rangle$ possess exactly one ebit entanglement [16] and the amount of entanglement is independent α. Dountlessly the ECS is a MES as it can be rewritten as

$$|\alpha;\alpha\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle_+\otimes|\alpha\rangle_- + |\alpha\rangle_-\otimes|\alpha\rangle_+\right) \tag{4}$$

in terms of the even and odd coherent states $|\alpha\rangle_\pm$. Eq.(4) shows that the state $|\alpha;\alpha\rangle$ manifestly has one ebit of entanglement.

In this paper we give conditions under which general bipartite entangled nonorthogonal states become MES. Using the conditions we construct a large class of bipartite maximally entangled nonorthogonal states in both the bipartite and multipartite systems. We also propose some methods to generate the bipartite maximally entangled nonorthogornal states.

II. MES CONDITION FOR BIPARTITE ENTANGLED STATES

We begin with a standard general bipartite entangled state [17,18]

$$|\Psi\rangle = \mu|\bar{\alpha}\rangle\otimes|\bar{\beta}\rangle + \nu|\bar{\gamma}\rangle\otimes|\bar{\delta}\rangle, \tag{5}$$

where $|\bar{\alpha}\rangle$ and $|\bar{\gamma}\rangle$ are *normalized states* of system 1 and similarly $|\bar{\beta}\rangle$ and $|\bar{\delta}\rangle$ are states of system 2 with complex μ and ν. We consider the nonorthogonal case, i.e., the overlaps $\langle\bar{\alpha}|\gamma\rangle$ and $\langle\bar{\beta}|\bar{\delta}\rangle$ are nonzero. After normalization, the bipartite state $|\Psi\rangle$ is given by

$$|\Psi\rangle = a|\bar{\alpha}\rangle\otimes|\bar{\beta}\rangle + d|\bar{\gamma}\rangle\otimes|\bar{\delta}\rangle, \tag{6}$$

where $a = \mu/N_{12}$, $d = \nu/N_{12}$, and

$$N_{12} = \sqrt{|\mu|^2 + |\nu|^2 + \mu\nu^*\langle\bar{\gamma}|\bar{\alpha}\rangle\langle\bar{\delta}|\bar{\beta}\rangle + \mu^*\nu\langle\bar{\alpha}|\bar{\gamma}\rangle\langle\bar{\beta}|\bar{\delta}\rangle}. \tag{7}$$

The two nonorthogonal states $|\bar{\alpha}\rangle$ and $|\bar{\gamma}\rangle$ are assumed to be linearly independent and span a two-dimensional

subspace of the Hilbert space. Then we choose an orthogonal basis $\{|\mathbf{0}\rangle, |\mathbf{1}\rangle\}$ as

$$|\mathbf{0}\rangle = |\bar{\alpha}\rangle, |\mathbf{1}\rangle = (|\bar{\gamma}\rangle - p_1|\bar{\alpha}\rangle)/N_1 \text{ for system 1,}$$
$$|\mathbf{0}\rangle = |\bar{\delta}\rangle, |\mathbf{1}\rangle = (|\bar{\beta}\rangle - p_2|\bar{\delta}\rangle)/N_2 \text{ for system 2,} \qquad (8)$$

where

$$p_1 = \langle\bar{\alpha}|\bar{\gamma}\rangle,\ N_1 = \sqrt{1-|p_1|^2},$$
$$p_2 = \langle\bar{\delta}|\bar{\beta}\rangle, N_2 = \sqrt{1-|p_2|^2}. \qquad (9)$$

Under these basis the entangled state $|\Psi\rangle$ can be rewritten as

$$|\Psi\rangle = (ap_2 + dp_1)|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle$$
$$+aN_2|\mathbf{0}\rangle \otimes |\mathbf{1}\rangle + dN_1|\mathbf{1}\rangle \otimes |\mathbf{0}\rangle, \qquad (10)$$

which shows that the general entangled nonorthogonal state is considered as a state of two logical qubits.

Then it is straightforward to obtain the reduced density matrix $\rho_{1(2)}$ and the two eigenvalues of ρ_1 are given by [18]

$$\lambda_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4|adN_1N_2|^2} \qquad (11)$$

which are identical to those of ρ_2. The corresponding eigenvectors of $\rho_{1(2)}$ is denoted by $|\pm\rangle_{1(2).}$ Then the general theory of the Schmidt decomposition [19] implies that the normalized state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = c_+|+\rangle_1 \otimes |+\rangle_2 + c_-|-\rangle_1 \otimes |-\rangle_2 \qquad (12)$$

with $|c_\pm|^2 = \lambda_\pm$.

From Eqs.(11) and (12) we immediately know that the condition for the state $|\Psi\rangle$ be a MES is $|2adN_1N_2| = 1$. Using Eqs.(7) and (9), we rewrite the condition explicitly as $\mathcal{C} = 1$, where

$$\mathcal{C} = \frac{2|\mu||\nu|\sqrt{(1-|p_1|^2)(1-|p_2|^2)}}{|\mu|^2 + |\nu|^2 + \mu\nu^* p_1^* p_2 + \mu^*\nu p_1 p_2^*}. \qquad (13)$$

Now we show that the quantity $\mathcal{C}$ is exactly one measure of entanglement, the concurrence [20] for two qubits. There are different measures of entanglement. One simple measure is the concurrence. Since the system 1 and 2 in the bipartite state (5) are essentially two-state systems, we can characterize the entanglement of bipartite state by the concurrence. The concurrence for a pure state $|\psi\rangle$ is defined by $\mathcal{C} = |\langle\psi|\sigma_y \otimes \sigma_y|\psi^*\rangle|$. Here $\sigma_y = i(|\mathbf{1}\rangle\langle\mathbf{0}| - |\mathbf{0}\rangle\langle\mathbf{1}|)$. A direct calculation shows that the concurrence of the bipartite state $|\Psi\rangle$ is just the quantity $\mathcal{C}$ given by Eq.(13). Then the condition for the state $|\Psi\rangle$ be a MES is that the concurrence of the state is equal to 1 as we hoped.

For orthorgonal state, $p_1 = p_2 = 0$, and the concurrence $\mathcal{C} = 2|\mu||\nu|/(|\mu|^2 + |\nu|^2)$ which obviously satisfies $0 \le \mathcal{C} \le 1$. The state $|\Psi\rangle$ becomes a MES when $|\mu| = |\nu| = 1$ as we expected. For partly orthorgonal state, $p_1 \neq 0, p_2 = 0$, Eq.(13) becomes

$$\mathcal{C} = 2|\mu||\nu|\sqrt{1-|p_1|^2}/(|\mu|^2 + |\nu|^2). \qquad (14)$$

Then the partly orthogonal state be a MES is when the inner product $p_1 = 0$. For completely nonorthorgonal state, $p_1 \neq 0$ and $p_2 \neq 0$. It is remarkable to see that we still have possibilities to make the concurrence $\mathcal{C}$ be 1. One case for $\mathcal{C} = 1$ is given by

$$\mu = -\nu,$$
$$\langle\bar{\alpha}|\bar{\gamma}\rangle = \langle\bar{\delta}|\bar{\beta}\rangle. \qquad (15)$$

The necessary and sufficient condition for the state $|\Psi\rangle$ to be a MES is discussed in detail in another paper [21]. We call Eq.(15) as the MES condition for the general state $|\Psi\rangle$. The MES condition (15) immediately gives a interesting antisymmetric MES

$$|\Psi_a\rangle = \frac{1}{\sqrt{2(1-|\langle\bar{\alpha}|\bar{\beta}\rangle|^2)}}\left(|\bar{\alpha}\rangle \otimes |\bar{\beta}\rangle - |\bar{\beta}\rangle \otimes |\bar{\alpha}\rangle\right). \quad (16)$$

The amount of entanglement of the state is exactly one ebit and the entanglement is independent of the parameters involved. However for a symmetric state

$$|\Psi_s\rangle = \frac{1}{\sqrt{2(1+|\langle\bar{\alpha}|\bar{\beta}\rangle|^2)}}\left(|\bar{\alpha}\rangle \otimes |\bar{\beta}\rangle + |\bar{\beta}\rangle \otimes |\bar{\alpha}\rangle\right), \quad (17)$$

the corresponding concurrence is $\mathcal{C} = \frac{1-|\langle\bar{\alpha}|\bar{\beta}\rangle|^2}{1+|\langle\bar{\alpha}|\bar{\beta}\rangle|^2}$, which indicates that the symmetric state is not maximally entangled except the orthorgonal case $\langle\bar{\alpha}|\bar{\beta}\rangle = 0$. Note that states $|\bar{\alpha}\rangle$ and $|\bar{\beta}\rangle$ are different normalized arbitrary states. From the above discussions we see that the relative phase plays an important role on the entanglement.

Hirota *et al.* [22] have found that the state $|\Psi_a\rangle$ is a MES. However they impose a restriction that the overlap $\langle\bar{\alpha}|\bar{\beta}\rangle$ is a real number. As we discussed here, this restriction is not necessary and the states $|\bar{\alpha}\rangle$ and $|\bar{\beta}\rangle$ can be arbitrary. As a illustration of the importance of complex overlap, we consider a state

$$|\alpha, \alpha^*\rangle = \frac{1}{\sqrt{2(1-|\langle\alpha|\alpha^*\rangle|^2)}}\left(|\alpha\rangle \otimes |\alpha^*\rangle - |\alpha\rangle \otimes |\alpha^*\rangle\right),$$

which is maximally entangled. The overlap $\langle\alpha|\alpha^*\rangle = e^{|\alpha|^2(e^{-i2\theta}-1)}$ is real only when $\alpha = |\alpha|e^{i\theta}$ is real or pure imaginary. So the MES with real overlap is a small subset of the set formed by the MES with complex overlap.

More maximailly entangled ECS can be constructed. For instance, the bosonic coherent state $|\alpha\rangle$ can be replaced by the abstract su(2) coherent state and su(1,1) coherent state in the state $|\alpha;\alpha\rangle$, and then obtain the corresponding su(2) and su(1,1) ECS [8] with one ebit of entanglement.

III. MORE GENERAL ENTANGLED NONORTHOGONAL STATES.

Now we consider a more general entangled coherent state of the following type

$$|\Phi\rangle = a|\bar{\alpha}\rangle \otimes |\bar{\beta}\rangle + b|\bar{\alpha}\rangle \otimes |\bar{\delta}\rangle + c|\bar{\gamma}\rangle \otimes |\bar{\beta}\rangle + d|\bar{\gamma}\rangle \otimes |\bar{\delta}\rangle \tag{18}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states. We assume that the state $|\Phi\rangle$ is a normalized state. When $b = c = 0$, the state $|\Phi\rangle$ reduces to the state $|\Psi\rangle$. One typical useful example of this type of states is the state generated by the interaction

$$H = \chi a_1^\dagger a_1 a_2^\dagger a_2, \tag{19}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators of system i. After an interaction time $t = \pi/\chi$, from the initial product of coherent states $|\alpha\rangle \otimes |\beta\rangle$, the output state is [23]

$$\begin{aligned}|\phi\rangle &= \frac{1}{2}\left[(|\alpha\rangle + |-\alpha\rangle) \otimes |\beta\rangle + (|\alpha\rangle - |-\alpha\rangle) \otimes |-\beta\rangle\right] \\ &= \frac{1}{2}(|\alpha\rangle \otimes |\beta\rangle + |\alpha\rangle \otimes |-\beta\rangle \\ &\quad +|-\alpha\rangle \otimes |\beta\rangle - |-\alpha\rangle \otimes |-\beta\rangle)\end{aligned} \tag{20}$$

This state was successfully used to construct entangled coherent–state qubits in an ion trap [24].

Using the same technique of above section we can consider the state $|\Phi\rangle$ as a two–qubit state. We write the state as the form of qubits

$$\begin{aligned}|\Phi\rangle &= (ap_2 + b + cp_1p_2 + dp_1)|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \\ &\quad +N_2(a + cp_1)|\mathbf{0}\rangle \otimes |\mathbf{1}\rangle \\ &\quad +N_1(d + cp_2)|\mathbf{1}\rangle \otimes |\mathbf{0}\rangle + cN_1N_2|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle.\end{aligned} \tag{21}$$

in the basis defined in Eq.(8).

For a general pure state

$$|\psi\rangle = a|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle + b|\mathbf{0}\rangle \otimes |\mathbf{1}\rangle + c|1\rangle \otimes |\mathbf{0}\rangle + d|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle, \tag{22}$$

the concurrence is given by [20]

$$C = 2|ad - bc|. \tag{23}$$

From Eqs.(21) and (23), the concurrence of the state $|\Phi\rangle$ is obtained as

$$C = 2\sqrt{1 - |p_1|^2}\sqrt{1 - |p_2|^2}|ad - bc|. \tag{24}$$

When $b = c = 0$ Eq.(24) reduces to Eq.(13) as we expected. From Eq.(24) the corresponding concurrence of the state $|\phi\rangle$ is simply obtained as

$$C = \sqrt{(1 - e^{-4|\alpha|^2})(1 - e^{-4|\beta|^2})}. \tag{25}$$

We see that the state becomes maximally entangled state if and only if the amplitudes $|\alpha| \gg 1$ and $|\beta| \gg 1$.

IV. BIPARTITE ENTANTGLEMENT IN MULTIPARTITE SYSTEMS

It is more interesting to ask if we can obtain bipartite MES in multipartite systems. A bipartite MES with even systems we can offer is

$$\begin{aligned}|\bar{\alpha}; \bar{\beta}\rangle_{2N} = &|\bar{\alpha}\rangle \otimes ... \otimes |\bar{\alpha}\rangle \otimes |\bar{\beta}\rangle \otimes ... \otimes |\bar{\beta}\rangle - \\ &|\bar{\beta}\rangle \otimes ... \otimes |\bar{\beta}\rangle \otimes |\bar{\alpha}\rangle \otimes ... \otimes |\bar{\alpha}\rangle\end{aligned} \tag{26}$$

up to a normalization constant. To see the fact that this state is a MES we consider the first N systems as system 1 and the other N systems as system 2. By this observation, these two states satisfy the MES condition (15), i.e., they are the MES in the sense that the concurrence $\mathcal{C}_{(12...N)(N+1,N+2...2N)}$between the first N systems and the second N systems is equal to one. Of course we can construct more complicated bipartite MES in the multipartite system according to the MES condition.

Now we consider a ECS defined by

$$|\alpha; -\alpha\rangle_N = |\alpha\rangle \otimes ... \otimes |\alpha\rangle - |-\alpha\rangle \otimes ... \otimes |-\alpha\rangle, \tag{27}$$

For even N, this state is a bipartite MES, however for odd N, usually it is not. The above state can be considered as multipartite maximally entangled states if the coherent states $|\pm\alpha\rangle$are considered as logical qubits as in Eq.(1).

After normalization, the MES $|\alpha; -\alpha\rangle_N$ is expanded as

$$\begin{aligned}|\alpha; -\alpha\rangle_N &= \frac{1}{\sqrt{2(1 - e^{-2N|\alpha|^2})}} \\ &\quad \times (|\alpha\rangle \otimes ... \otimes |\alpha\rangle - |-\alpha\rangle \otimes ... \otimes |-\alpha\rangle) \\ &= \frac{1}{\sqrt{\sinh(N|\alpha|^2)}} \\ &\quad \sum_{n_1...n_N}^{\infty} \frac{\alpha^{n_1+...+n_N}\left[1 - (-1)^{n_1+...+n_N}\right]}{2\sqrt{n_1!...n_N!}} \\ &\quad \times|n_1...n_N\rangle\end{aligned} \tag{28}$$

where $|n_1...n_N\rangle = |n_1\rangle \otimes ... \otimes |n_N\rangle$ and $|n_k\rangle$ are Fock states of system k.

In the limit $|\alpha| \to 0$, we see that only the terms with $n_1 + ... + n_N = 1$ survive, and the resultant state is

$$\begin{aligned}|\mathrm{W}\rangle_N = \frac{1}{\sqrt{N}}&(|100...0\rangle + |0100...0\rangle + ... \\ &+|0000...1\rangle).\end{aligned} \tag{29}$$

It is interesting to see that the state is the so-called W state [25,26]. The entanglement of W state is maximally robust under disposal of any one of the qubits.

The state $|\alpha; -\alpha\rangle_N$ with even N is a bipartite MES and then the W state with even N is also a bipartite MES. For instance $|\mathrm{W}\rangle_4$ can be rewritten as

$$|W\rangle_4 = \frac{1}{\sqrt{2}}(|\Psi^+\rangle \otimes |00\rangle + |00\rangle \otimes |\Psi^+\rangle), \tag{30}$$

which manifestly has one ebit of entanglement. Here the state $|\Psi^+\rangle$ represents one of the Bell state, i.e,

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).$$

However the state $|\alpha; -\alpha\rangle_N$ with odd N is not a bipartite MES. For instance, from Eq.(23) the concurrence between system 1 and systems 2 and 3 of the state $|\alpha; -\alpha\rangle_3$ is obtained as

$$\mathcal{C}_{1(23)} = \frac{\sqrt{(1-e^{-4|\alpha|^2})(1-e^{-8|\alpha|^2})}}{(1-e^{-6|\alpha|^2})}. \tag{31}$$

In the limit $|\alpha| \to \infty$, the concurrence becomes 1 as we expected, and in the limit $|\alpha| \to 0$, the concurrence $\mathcal{C}_{1(23)} = \frac{2\sqrt{2}}{3}$, which can be understood as follows. The state $|\alpha; -\alpha\rangle_3$ becomes W state in the limit $|\alpha| \to 0$, and for state $|\mathrm{W}\rangle_N$, an equality [27]

$$\mathcal{C}_{12}^2 + \mathcal{C}_{13}^2 + ... + \mathcal{C}_{1N}^2 = \mathcal{C}_{1(23..n)}^2 \tag{32}$$

holds. For $|\mathrm{W}\rangle_3$ the concurrence $\mathcal{C}_{12} = \mathcal{C}_{13} = 2/3$, therefore $\mathcal{C}_{1(23)} = \frac{2\sqrt{2}}{3}$.

In three-qubit system we can construct a bipartite MES as

$$\begin{aligned}|\alpha; \frac{\alpha}{\sqrt{2}}\rangle_3 = |\alpha\rangle \otimes |\frac{\alpha}{\sqrt{2}}\rangle \otimes |\frac{\alpha}{\sqrt{2}}\rangle \\ -|-\alpha\rangle \otimes |-\frac{\alpha}{\sqrt{2}}\rangle \otimes |-\frac{\alpha}{\sqrt{2}}\rangle,\end{aligned} \tag{33}$$

In the limit $|\alpha| \to 0$, it reduces to the MES $\frac{1}{\sqrt{2}}(|1\rangle \otimes |00\rangle + |0\rangle \otimes |\Psi^+\rangle)$. Further in odd systems the bipartite MES is constructed as

$$\begin{aligned}|\alpha; \frac{\alpha}{\sqrt{2N}}\rangle_{2N+1} = |\alpha\rangle \otimes |\frac{\alpha}{\sqrt{2N}}\rangle \otimes ... \otimes |\frac{\alpha}{\sqrt{2N}}\rangle \\ -|-\alpha\rangle \otimes |\frac{-\alpha}{\sqrt{2N}}\rangle \otimes ... \otimes |\frac{-\alpha}{\sqrt{2N}}\rangle\end{aligned} \tag{34}$$

with the concurrence $\mathcal{C}_{1(23..2N+1)} = 1$, which results from the identity

$$\langle\alpha| - \alpha\rangle = (\langle\alpha/\sqrt{2N}| - \alpha/\sqrt{2N}\rangle)^{2N} \tag{35}$$

and the MES condition.

V. GENERATION OF THE ENTANGLED STATES

Now we consider how to generate the bipartite maximally entangled nonorthogonal states in both bipartite and multipartite systems.

A. Bipartite entangled states

One method is already given by Barenco *et al.* [28] and Bužek and Hillery [29], and based on controlled-SWAP gate which is described by the following transformation

$$\begin{aligned}|0\rangle|\bar{\alpha}\rangle|\bar{\beta}\rangle \to |0\rangle|\bar{\alpha}\rangle|\bar{\beta}\rangle, \\ |1\rangle|\bar{\alpha}\rangle|\bar{\beta}\rangle \to |1\rangle|\bar{\beta}\rangle|\bar{\alpha}\rangle.\end{aligned} \tag{36}$$

Let the input state of the controlled-SWAP gate is $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|\bar{\alpha}\rangle|\bar{\beta}\rangle$ and we measure the output state. If we measure the qubit on the state $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, we obtain exactly the antisymmetric maximally entangled state $|\Psi_a\rangle$ (23). The entanglement swapping method [30] can be used to generate entangled coherent states in trapped-ion systems [31,32], which is also discussed in Ref. [33]. Here we generalize the method proposed by van Enk and Hirota [16], who have studied how to generate $|\alpha; \alpha\rangle$ by 50/50 beam splitter.

The 50/50 beam splitter is described by $B_{1,2} = e^{i\frac{\pi}{4}(a_1^\dagger a_2 + a_2^\dagger a_1)}$, which transforms the state $|\alpha\rangle_1 \otimes |\beta\rangle_2$ as

$$\begin{aligned}&B_{1,2}|\alpha\rangle_1 \otimes |\beta\rangle_2 \\ &= |(\alpha + i\beta)/\sqrt{2}\rangle_1 \otimes |(\beta + i\alpha)/\sqrt{2}\rangle_2.\end{aligned} \tag{37}$$

Further using the phase shifter $P_2 = e^{-i\frac{\pi}{2}a_2^\dagger a_2}$ which makes phase shifting by $-\pi/2$, we can have the transformation $\mathcal{B}_{1,2} = P_2 B_{12} P_2$, which transforms the coherent states as

$$\mathcal{B}_{1,2}|\alpha\rangle_1 \otimes |\beta\rangle_2 = |\epsilon_+\rangle_1 \otimes |\epsilon_-\rangle_2, \tag{38}$$

where $\epsilon_\pm = (\alpha \pm \beta)/\sqrt{2}$. Now let the input state be $|\alpha\rangle_{1-} \otimes |\beta\rangle_2$, i.e., the input state is the direct product of the odd coherent state $|\alpha\rangle_{1-}$ and the coherent state $|\beta\rangle_2$. After the transformation $\mathcal{B}_{1,2}$, we obtain the output state as

$$|\epsilon_+\rangle_1 \otimes |\epsilon_-\rangle_2 - |-\epsilon_-\rangle_1 \otimes |-\epsilon_+\rangle_2 \tag{39}$$

up to a normalization constant. Apply another phase shifter $e^{-i\pi a_2^\dagger a_2}$ on the above state, we obtain the unnormalized state

$$|\epsilon_+\rangle_1 \otimes |-\epsilon_-\rangle_2 - |-\epsilon_-\rangle_1 \otimes |\epsilon_+\rangle_2, \tag{40}$$

which is exactly of the form of $|\Psi_a\rangle$(16). So the two-parameter ECS is a MES independent of the two parameters $\epsilon_\pm$.

From the above procedure we can see that the odd coherent state plays an important role. If we replace the odd coherent state by the even coherent state and repeat the procedure, the resultant state is not a MES. If we let the input state be the product state of two odd coherent states, $|\alpha\rangle_{1-} \otimes |\alpha\rangle_{2-}$, the resultant state is given by

$$|\sqrt{2}\alpha\rangle_{1+} \otimes |0\rangle_2 - |0\rangle_1 \otimes |\sqrt{2}\alpha\rangle_{2+}, \tag{41}$$

which is also a MES. If we replace the input state $|\alpha\rangle_{1-} \otimes |\alpha\rangle_{2-}$ by $|\alpha\rangle_{1+} \otimes |\alpha\rangle_{2+}$ or $|\alpha\rangle_{1-} \otimes |\alpha\rangle_{2+}$, the resultant states are not MES.

B. Multipartite entangled coherent states

We first introduce the Kerr transformation

$$\mathcal{K} = \exp[-i\pi(a_i^\dagger a_i)^2/2], \tag{42}$$

where a_i and $a_i^\dagger$ are the annihilation and creation operators of the field mode i, respectively. It is well know that $\mathcal{K}$ can transfer a coherence state $|\alpha\rangle_i$ into a superposition of two coherent states [13] $|\pm\alpha\rangle_i$, i.e.,

$$\mathcal{K}_i|\alpha\rangle_i = 2^{-1/2}(|\alpha\rangle_i + i|-\alpha\rangle_i) \tag{43}$$

up to a trivial global phase. This superposition state has proved to be very useful in constructing optical analogs to Schrödinger's cat state [13]. Now we try to generate the following multipartite entangled coherent state with N modes

$$\begin{aligned}|\alpha;-\alpha\rangle_N^{'} = 2^{-1/2}(&|\alpha\rangle_1\otimes|\alpha\rangle_2\cdots|\alpha\rangle_N \\ &+i|-\alpha\rangle_1\otimes|-\alpha\rangle_2\cdots|-\alpha\rangle_N).\end{aligned} \tag{44}$$

This state is in fact a multipartite entangled state for continuous variables. van Loock and Braunstein [34] have used an interesting unitary transformation $\mathcal{U}$ to creat multipartite entangled states of continous variables. And later the transformation is used to realize optical cloning machine for coherent states [35]. The transformation $\mathcal{U}$ is defined as a quantum network of beam splitters as

$$\begin{aligned}\mathcal{U}_N = &\mathcal{B}_{N-1,N}^{'}\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)\mathcal{B}_{N-2,N-1}^{'}\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) \\ &\times\cdots\times\mathcal{B}_{1,2}^{'}\left(\sin^{-1}\frac{1}{\sqrt{N}}\right),\end{aligned} \tag{45}$$

where $\mathcal{B}_{i-1,i}^{'}(\theta) = \exp[\theta(a_{i-1}^\dagger a_i - a_i^\dagger a_{i-1})]$ is also a beam splitter transformation acting on mode $i-1$ and i. The action of the beam splitter on the two modes can be expressed as

$$\begin{aligned}&\mathcal{B}_{i-1,i}^{'}(\theta)\begin{pmatrix} a_{i-1}^\dagger \\ a_i^\dagger \end{pmatrix}\mathcal{B}_{i-1,i}^{'\dagger}(\theta) \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a_{i-1}^\dagger \\ a_i^\dagger \end{pmatrix},\end{aligned} \tag{46}$$

Directly from Eqs.(45) and (46), an useful relation is obtained as

$$\mathcal{U}_N\sqrt{N}a_1^\dagger\mathcal{U}_N^\dagger = \sum_{i=1}^N a_i^\dagger. \tag{47}$$

Eq.(47) leads to

$$\mathcal{U}_N D_1\left(\sqrt{N}\alpha\right)\mathcal{U}_N^\dagger = \prod_{i=1}^N D_i(\alpha), \tag{48}$$

where $D_i(\alpha) = \exp(\alpha a_i^\dagger - \alpha^* a_i)$ is the displacement operator for mode i.

Let the intial state of the N–mode systems be $|\sqrt{N}\alpha\rangle_1\otimes|0\rangle_2\cdots|0\rangle_N$, and let the unitary operator $\mathcal{U}_N\mathcal{K}_1$ act on it. Then we obtain

$$|\alpha;-\alpha\rangle_N^{'} = \mathcal{U}_N\mathcal{K}_1|\sqrt{N}\alpha\rangle_1\otimes|0\rangle_2\cdots|0\rangle_N. \tag{49}$$

That is to say, we can generate the multipartite ECS from a product N–mode state by successive application of the unitary operators $\mathcal{K}_1$ and $\mathcal{U}_N$. The proposed scheme is simple and can be in principle realized by the present technology.

We can also generate the multipartite W state [25,26]. From Eq.(47) we obtain

$$|\mathrm{W}\rangle = \mathcal{U}_N|100...0\rangle. \tag{50}$$

That is to say, the W state can be directly generated by applying the unitary operator $\mathcal{U}_N$ to the state $|100...0\rangle$.

Now let us see how to produce another kind of multipartite ECS. Let the initial state of N bosonic systems be

$$|\Psi_0\rangle = (|\alpha\rangle_1 - |-\alpha\rangle_1)\otimes|0\rangle_2\otimes|0\rangle_3\otimes...\otimes|0\rangle_N. \tag{51}$$

By applying the transformation $\mathcal{B}_{N-1,N}...\mathcal{B}_{3,4}\mathcal{B}_{1,2}$ to the the initial state, we obtain

$$\begin{aligned}&\mathcal{B}_{N-1,N}...\mathcal{B}_{3,4}\mathcal{B}_{1,2}|\Psi_0\rangle \\ &= |\frac{\alpha}{2^{1/2}}\rangle_1\otimes|\frac{\alpha}{2^1}\rangle_2\otimes...\otimes|\frac{\alpha}{2^{i/2}}\rangle_i\otimes... \\ &\quad\otimes|\frac{\alpha}{2^{(N-2)/2}}\rangle_{N-2}\otimes|\frac{\alpha}{2^{(N-1)/2}}\rangle_{N-1}\otimes|\frac{\alpha}{2^{(N-1)/2}}\rangle_N \\ &\quad-|\frac{-\alpha}{2^{1/2}}\rangle_1\otimes|\frac{-\alpha}{2^1}\rangle_2\otimes...\otimes|\frac{-\alpha}{2^{i/2}}\rangle_i\otimes... \\ &\quad\otimes|\frac{-\alpha}{2^{(N-2)/2}}\rangle_{N-2}\otimes|\frac{-\alpha}{2^{(N-1)/2}}\rangle_{N-1}\otimes|\frac{-\alpha}{2^{(N-1)/2}}\rangle_N.\end{aligned} \tag{52}$$

It is easy to check that the $\mathcal{C}_{1(23..N)} = 1$ due to the identity

$$\langle\frac{\alpha}{2^{1/2}}|\frac{-\alpha}{2^{1/2}}\rangle = \langle\frac{\alpha}{2^{(N-1)/2}}|\frac{-\alpha}{2^{(N-1)/2}}\rangle\prod_{i=2}^{N-1}\langle\frac{\alpha}{2^{i/2}}|\frac{-\alpha}{2^{i/2}}\rangle. \tag{53}$$

So this state is a bipartite MES. Note that here the integer N can be either even or odd.

VI. CONCLUSION

In conclusion we have given conditions under which general bipartite entangled nonorthogonal states become MES. According to the conditions a large class of bipartite maximally entangled nonorthogonal states are constructed in both the bipartite and multipartite systems.

A remarkable property of these MES is that the amount of entanglement are independent of parameters involved in the states. We also propose some methods to generate the MES. Specifically the multipartite entangled coherent states and the multipartite W state are generated by quantum networks of beam splitters.

The applications of the bipartite MES discussed in this paper are already considered in the context of quantum teleportation of coherent states [16] and entangled coherent states [36]. The MES are expected to have more applications in the quantum information processings. Throughout the paper we only consider the bipartite entanglement. The more difficult task is to quantify the genuine multipartite entanglement [25,37] in the multipartite nonorthogonal states, which are now under consideration.

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